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Lorentz transformations in bifurcating systems

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The linear response of a driven dissipative system near a bifurcation instability is studied. Using a previously established scaling relation, it is shown that the response to a small coherent signal can be characterized by a Lorentz transformation of a spinor whose components are the two orthogonal phases (quadratures) of the signal. The analysis shows that by forming an array of such systems it should be possible to demonstrate features of the Lorentz group experimentally, e.g., Wigner rotations and Berry's phase of the Lorentz group, in simple bifurcating systems.

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Lorentz transformations and the associated Lorentz group are usually considered as synonymous with the special theory of relativity. However, the Lorentz group (LG) is a mathematical group and one should therefore not be surprised to encounter the LG outside the special theory of relativity. For example, it has been known for a number of years that this group also enters into the theory of squeezed light [1–3]. This opens up for the possibility of studying features of the LG in nonrelativistic systems entirely outside the usual domain of the special theory of relativity; features which have not yet been observed experimentally neither in the special theory of relativity nor in squeezed light [2,3]. We mention, for example, the measurement of finite Wigner rotations (since Lorentz transformations in nonparallel directions do not commute but involve a rotation [2,4,5]), or Berry's geometrical phase associated with the LG [3,6].

In this paper we show that important features of the LG are also encountered in a driven dissipative dynamical system in the vicinity of a bifurcation point. The system under consideration is a nonlinear driven oscillator with linear damping. As an example we perturb the system with a small signal in the linear response limit and show that the response corresponds to a Lorentz boost of a spinor characterizing the input signal. The basis of our analysis is the existence of scaling properties in the vicinity of a period-doubling bifurcation point [7], which enables us to present an analytical solution of the transformation properties and moreover implies that these Lorentz-like transformations are in fact a general feature of a whole class of nonlinear bifurcating oscillators. Since the system under consideration is dissipative and therefore an open system, there are modifications to the standard Lorentz transformation, but, surprisingly, important features of the Lorentz group are preserved. As a result it should be possible from an experimental point of view to demonstrate properties of the Lorentz group; e.g., Wigner rotations and the geometrical phase of the Lorentz group, in simple bifurcating systems.

We consider a class of nonlinear damped oscillators described by the following second-order differential equation:

$$x_{tt} + \alpha x_t + \frac{dV(x)}{dx} = A_D \cos(2\omega_R t + \phi) + A_S \cos(\omega_R t + \theta). \qquad (1)$$

Here x is the position and $V(x)$ a generic nonlinear potential. The only restrictions on the potential are first, that the second-order derivative $d^2V(x)/dx^2|_{x=0} = \omega_0^2 > 0$. This means that the system has a small-amplitude relaxation-frequency given by $\omega_R = {\omega_0}^2 - \alpha^2/4$, where α is the damping constant with the constraint $\alpha \ll \omega_0$. Second, that the third-order derivative $d^3V(x)/dx^3|_{x=0} = \gamma \neq 0$. The oscillator is driven by a harmonic force of amplitude A_D, frequency $2\omega_R$, and phase ϕ. The second term on the left-hand side is a perturbing signal with amplitude A_S, frequency ω_R, and phase θ, and it is the system's response to this signal that will be studied here. The theory presented is based on the following assumptions. (i) As the driving amplitude A_D is increased the system undergoes a period-doubling bifurcation, and it is only the response of the system up till the bifurcation point that is studied. (ii) We drive the system at twice the resonance frequency, i.e., $\omega_D = 2\omega_R$. (iii) The coupling of the system to the environment can be characterized by a linear damping term, and the obtained results are valid in the limit $\alpha \to 0$.

The nonlinear term associated with γ breaks the symmetry of the potential around $x = 0$, and is responsible for the ability of the system to undergo a period-doubling bifurcation. Higher-order terms of the potential $V(x)$ become unimportant since the bifurcation point and thereby the limit cycle $x(t)$ tend to zero in the limit $\alpha \to 0$ studied here [7]. Therefore the class of nonlinear potentials covered by this theory is very broad.

On the basis of the above assumptions Eq. (1) can be solved in the limit of a linear response. First, Eq. (1) is linearized about the cycle $x_0(t)$ of the unperturbed system, i.e., for $A_S = 0$, which gives

$$\xi_{tt} + \alpha\xi_t + \left.\frac{d^2V}{dx^2}\right|_{x_0} \xi = 0 \,, \qquad (2)$$

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where $\xi(t) = x(t) - x_0(t)$ is the deviation from the limit cycle. In the limit of small damping the solution to the above homogeneous equation is [7]

$$\xi_\pm(t) = [\cos(\omega_R t + \phi/2) \pm \sin(\omega_R t + \phi/2)] \times \exp\left[\left(-1 \pm \frac{A_D}{A_C}\right)\frac{\alpha}{2}t\right], \tag{3}$$

where the critical value of the drive A_C (the bifurcation point) is given by

$$A_C = \frac{6\alpha V''(x_m)^{3/2}}{V'''(x_m)} = \frac{6\alpha\omega_0{}^3}{\gamma}. \tag{4}$$

Here x_m is the minimum of the potential. In the last expression we have used that $\omega_R \to \omega_0$ for $\alpha \to 0$. The above solution has been obtained by using the following expression for the limit cycle, $x_0(t) = [-A_D/(3\omega_R{}^2)]\cos(2\omega_R t+\phi)$, and only retaining terms in the potential $V(x)$ until third order. The theory is only valid for $A_D < A_C$, a restriction which indicates that the amplitude of the limit cycle scales with the damping. This indicates that the behavior of the system is controlled by the properties of the nonlinear potential near the minimum point. We note that the above result shows that in the scaling limit $\alpha \to 0$ the response has universal features.

By use of the above solutions [Eq. (3)] the system's response to the small signal $A_S\cos(\omega_R t+\theta)$ can be found as

$$\xi(t) = \xi_+(t)\int_{-\infty}^{t}\frac{\xi_-\,\sigma(s)}{W(s)}\,ds - \xi_-(t)\int_{-\infty}^{t}\frac{\xi_+\,\sigma(s)}{W(s)}\,ds\ . \tag{5}$$

Here $\sigma(t) = A_S\cos(t+\theta)$ is the small input signal and $W(s)$ the Wronski determinant given by $W(t) = \omega_R\exp(-\alpha t)$. Solving this equation we obtain the response,

$$\xi(t) = \frac{A_S}{\alpha\,\omega_R}\left[\frac{1}{1-\lambda^2}\sin(\omega_R t+\theta) + \frac{\lambda}{1-\lambda^2}\cos(\omega_R t+\phi-\theta)\right]\ , \tag{6}$$

where $\lambda = A_D/A_C$. The response to higher harmonics than the resonance frequency has been neglected since it can be shown that they vanish in the limit $\alpha \to 0$.

We now turn to a representation of the above results in terms of a transformation of a two-component spinor. We first note that the input signal $\sigma(t)$ can be viewed as the projection of a constant spinor $\boldsymbol{\xi}$ projected onto a rotating unit vector $\mathbf{e}(t)$:

$$\sigma(t) = A_S\cos(\omega_R t+\theta) = \boldsymbol{\xi}\cdot\mathbf{e}(t), \tag{7}$$

where

$$\boldsymbol{\xi} = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = A_S\begin{pmatrix}\cos\theta\\ -\sin\theta\end{pmatrix}\ ,\qquad \mathbf{e}(t) = \begin{pmatrix}\cos\omega_R t\\ \sin\omega_R t\end{pmatrix}. \tag{8}$$

As a result, the response in Eq. (6) can be expressed as a linear mapping of the constant two-component spinor (ξ_1,ξ_2), followed by a projection on the rotating unit vector $\mathbf{e}(t)$. The linear transformation of the two components (ξ_1,ξ_2) is given by

$$\begin{pmatrix}\xi_1'\\ \xi_2'\end{pmatrix} = \mathbf{\Lambda}(\lambda,\phi,A_S,\alpha,\omega_R)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \tag{9}$$

where the transformation matrix takes the form

$$\mathbf{\Lambda}(\lambda,\phi,A_S,\alpha,\omega_R) = \left(\frac{A_S}{\alpha\,\omega_R}\right)\begin{pmatrix}\dfrac{\lambda}{1-\lambda^2}\cos\phi & -\dfrac{1}{1-\lambda^2}-\dfrac{\lambda}{1-\lambda^2}\sin\phi\\ \dfrac{1}{1-\lambda^2}-\dfrac{\lambda}{1-\lambda^2}\sin\phi & -\dfrac{\lambda}{1-\lambda^2}\cos\phi\end{pmatrix}. \tag{10}$$

The determinant of the matrix is $\det\mathbf{\Lambda} = A_S/[\alpha\,\omega_R(1-\lambda^2)]$. The above transformation matrix can be expressed in a more transparent form by means of the Pauli matrices

$$\sigma_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{11}$$

and the parametrization

$$\cosh\Phi/2 = \frac{1}{(1-\lambda^2)^{1/2}}\ ,\qquad \sinh\Phi/2 = \frac{\lambda}{(1-\lambda^2)^{1/2}}\ . \tag{12}$$

In exponential form we finally obtain

$$\mathbf{\Lambda}(\Phi,\phi) = -i\sigma_y\ \exp\left(-\frac{\boldsymbol{\sigma}\cdot\mathbf{n}}{2}\,\Phi\right)\,a_0\,\cosh\frac{\Phi}{2}, \tag{13}$$

where $\boldsymbol{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$, $\mathbf{n} = (\cos\phi, 0, \sin\phi)$, and $a_0 = A_S/(\alpha\,\omega_R)$. The first Pauli matrix $(-i\sigma_y)$ expresses a rotation of $-\pi/2$ about the "y axis" and can be related to the phase shift of the input signal at resonance. The second part corresponds to a Lorentz transformation of a spinor [8]. As is well known, such a Lorentz transformation involves either a contraction or a stretching (the points on a circle are transformed into an ellipse). This feature reflects the properties of the bifurcation in the following sense. As the bifurcation point is approached, one direction in phase space becomes critical (an eigenvalue tends to -1), and as a result this direction in phase space is stretched. The orthogonal direction in phase space is governed by the second eigenvalue, which at the same time is stabilized, with the result that this direction is contracted. In this theory the bifurcation point A_C plays a role similar to the speed of light in the special theory of relativity. We note that the second component of the directional vector $\mathbf{n}$ is always zero. This indicates that the present transformation is a subgroup of the full Lorentz group, corresponding to one less spatial dimension. The direction of the Lorentz boost is given by the phase ϕ of the driving field. Finally, the last factor is a scalar function that makes the transformation an expanding one, i.e., $\det\mathbf{\Lambda}(\Phi,\phi) \geq 1$. This reflects the fact that we are dealing with an open system being pumped by the driving field.

The transformation in Eq. (13) constitutes our main result. Considering now an array of identical nonlinear oscillators driven at different amplitudes and phases and coupled in such a way that the output of one oscillator becomes the input of the next one, it should be possible to demonstrate specific features of the LG. For example, two coupled oscillators correspond to the successive transformations

$$\mathbf{\Lambda}(\Phi_2,\phi_2)\ \ \mathbf{\Lambda}(\Phi_1,\phi_1) = -a_0^2 \cosh\frac{\Phi_2}{2}\cosh\frac{\Phi_1}{2}\exp\left(\frac{\boldsymbol{\sigma}\cdot\mathbf{n}_2}{2}\Phi_2\right) \times\exp\left(-\frac{\boldsymbol{\sigma}\cdot\mathbf{n}_1}{2}\Phi_1\right), \tag{14}$$

where $\mathbf{n}_i = (\cos\phi_i, 0, \sin\phi_i)$, and where the effect of σ_y has been incorporated. Disregarding the prefactor this corresponds to two successive Lorentz transformations. As is well known from Wigner's work [4] the combination of two Lorentz transformations in nonparallel directions corresponds to a single Lorentz transformation and an additional spatial rotation, the so-called Wigner rotation. Consequently, the above transformation has the general form

$$\mathbf{\Lambda}(\Phi_2,\phi_2)\ \ \mathbf{\Lambda}(\Phi_1,\phi_1) = -a_0^2 \cosh\frac{\Phi_2}{2}\cosh\frac{\Phi_1}{2}\exp\left(i\,\frac{\sigma_y\,\Delta}{2}\right) \times\exp\left(-\frac{\boldsymbol{\sigma}\cdot\mathbf{n}_3}{2}\Phi_3\right), \tag{15}$$

where Δ is the Wigner rotation. In the special case where $\mathbf{n}_1 = (1,0,0)$ and $\mathbf{n}_2 = (0,0,1)$ the rotation angle is given by

$$\tan\frac{\Delta}{2} = -\tanh\frac{\Phi_1}{2}\tanh\frac{\Phi_2}{2}, \tag{16}$$

and the parameters in the Lorentz transformation become $\mathbf{n}_3 = (\cos\phi_3, 0, \sin\phi_3)$, where

$$\cosh\Phi_3 = \cosh\Phi_1\cosh\Phi_2, \qquad \tan\phi_3 = -\tanh\Phi_2\ /\ \sinh\Phi_1. \tag{17}$$

The above result clearly demonstrates that it is possible to find features of the LG in these simple nonlinear systems. Using a system which undergoes a period-doubling bifurcation is very convenient, since the system's response to the small signal will be at the frequency ω_R whereas the driving frequency will be at $2\omega_R$. So by the use of a filtering process it should be possible to isolate the response. The main difference between the LG and a series of transformations of the above type is the scalar function that makes the transformation expand. In an experimental setup this feature can be removed by an appropriate attenuation of the signal after each transformation.

In conclusion it has been shown that a large class of nonlinear oscillators in the vicinity of a bifurcation point have transformation properties which are similar to Lorentz transformations in 2+1 dimensions. For example, the response of an oscillator to a small harmonic signal can be viewed as a Lorentz transformation of a spinor. The systems studied here are by nature dissipative and the transformation properties are therefore intimately connected with this fact, and it follows that successive transformations do not form a mathematical group since the inverse of a transformation does not exist. Nevertheless, many features of the LG are still preserved, like the Wigner rotations caused by noncommuting Lorentz transformations. It is important to realize that the role of dissipation is not restricted to the influence on the transformation properties. The damping in effect enters in a crucial way everywhere in our analysis. For example, if there were no damping the transformations described here would not exist. This is clear from the structure of the solutions in (3) to the homogeneous equation since, if the damping is zero, one of the solutions will be unstable, and the theory breaks down. The theory also ceases to be valid at the bifurcation point, since the linear response diverges. In real systems such a divergence does not take place due to nonlinearities, but in many cases deviations only occur very close to the bifurcation point. The linear theory is therefore expected to work well, except at a small region close to the bifurcation point.

The difference between the present approach and the theory presented in Refs. [2,3] on the LG and squeezed states, can be stated as follows: The present theory is in contrast to the theory on squeezed states based on dissipative systems, a dissipation which is of importance for the existence of the Lorentz-like transformations discussed here. In addition, the present theory is formulated in a scaling limit, which makes the results universal for a large class of systems.

Although the theory is formulated in the limit of damping tending to zero (at the transition between a dissipative system and a Hamiltonian system), we believe that features of the theory remain valid at finite values of the damping. It is therefore hoped that it should be possible to demonstrate properties of the Lorentz group experimentally in simple nonlinear oscillators, such as Wigner rotations and the geometrical phase of the Lorentz group.

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[1] A. O. Barut and L. Giradello, Commun. Math. Phys. **21**, 41 (1971); A. Peremolov, *ibid.* **26**, 22 (1972); *Generalized Coherent States and Their Applications* (Springer-Verlag, Heidelberg, 1986).
[2] D. Han, Y. S. Kim, and M. E. Noz, Phys. Rev. A **37**, 807 (1988).
[3] R. Y. Chiao and T. F. Jordan, Phys. Lett. **132**, 77 (1988).
[4] E. P. Wigner, Ann. Math. **40**, 149 (1939).
[5] Thomas precession is a physical example of the time rate of Wigner rotations, and enters in the spin-orbit coupling in atomic physics [L. H. Thomas, Nature (London) **117**, 514 (1926)].
[6] M. V. Berry, Proc. R. Soc. London Ser. A **392**, 45 (1984).
[7] K. Flensberg and H. Svensmark, Phys. Rev. E **47**, 2190 (1993).
[8] See, for example, Lewis H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1985), p. 40.